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Short Course @ISAP2010
in MACAO

Eigenvalues and Eigenvectors in Array Antennas

Optimization of Array Antennas for High Performance

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Self-introduction

- **Nobuyoshi Kikuma**
  - Professor
  - Research interests:
    - adaptive and signal processing array
    - multipath propagation analysis
    - mobile and indoor wireless communication
    - wireless power transmission
  - Hobby:
    - MUSIC (karaoke, etc).

On this short course

- Spatial signal processing technology using array antenna has been one of the important approaches for improving the performance of communications and radars.
- This short course expresses the optimization of the array antenna for its high performance in various applications.

On this short course(cont'd)

- Significance of eigenvalues and eigenvectors of various matrices used in the array antenna is explained.
- Key words:
  - Gain of the array antenna
  - Optimum weights of the adaptive array
  - Array weights for direction-of-arrival (DOA) estimation (e.g. MUSIC)

Ref.: D.K. Cheng, "Optimization techniques for antenna array," Proc. IEEE, vol.59, No.12, pp.1664-1674, Dec. 1971.

Applications of radio waves

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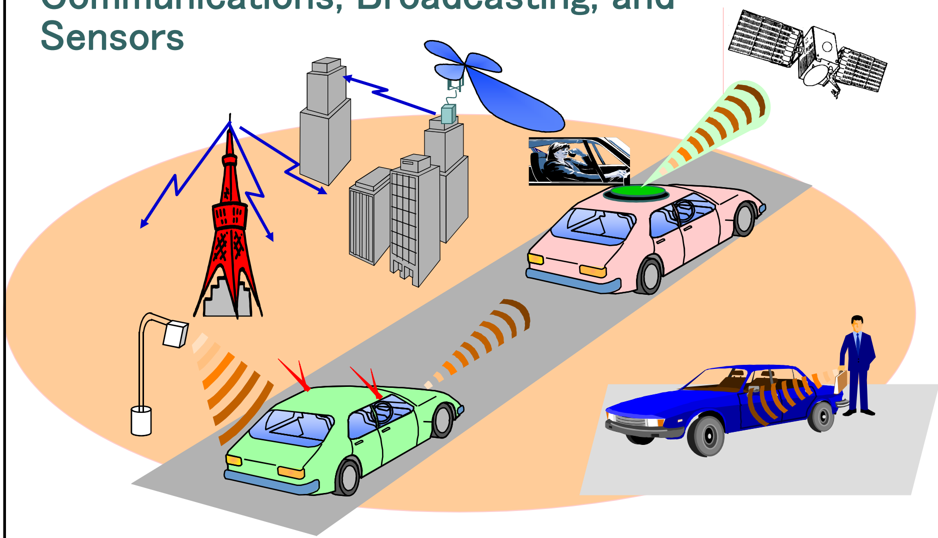
Communications, Broadcasting, and Sensors



Antennas are important components.

Radio Propagation in Mobile Communications

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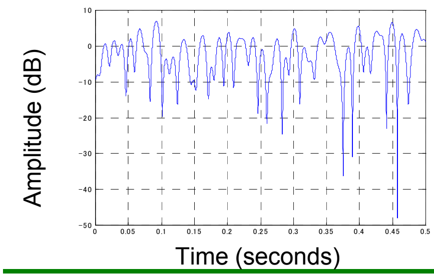
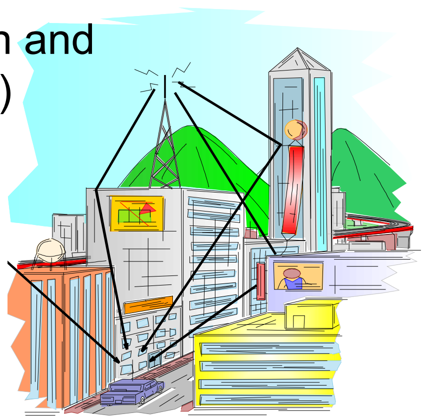
- No line of sight
- Reflection, diffraction and scattering (multipath)
- Heavy fading





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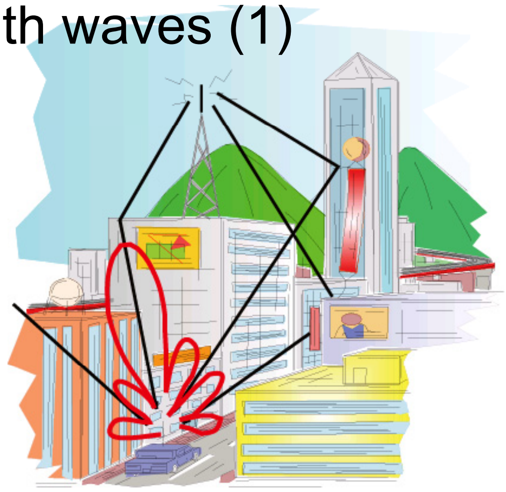
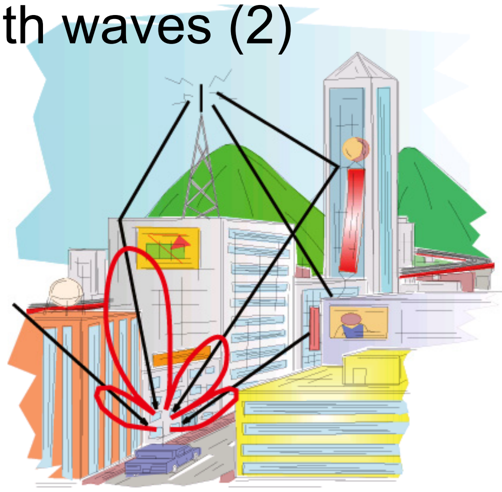
Example of antenna pattern for multipath waves (1)



Only one dominant wave is received.

Example of antenna pattern for multipath waves (2)



All multipath waves are combined appropriately.

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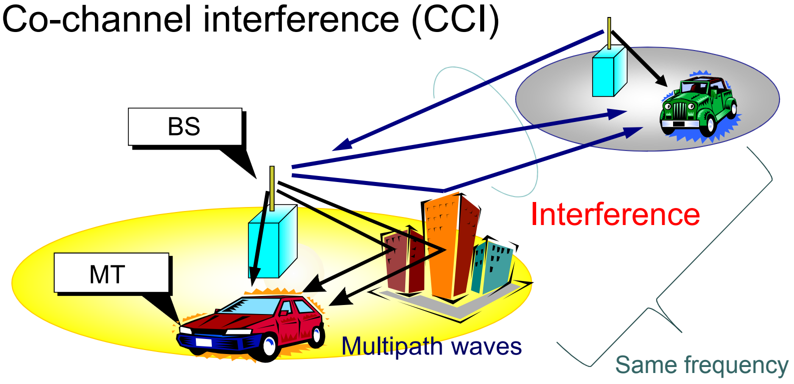
Radio Propagation in Cellular Communication Systems

- Effective utilization of frequency resources
- Co-channel interference (CCI)



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Techniques expected in antenna systems

- Fading recovery
- Suppressing or Canceling CCI

Control of directional pattern using array antenna:

Adaptive array signal processing (pattern optimization)

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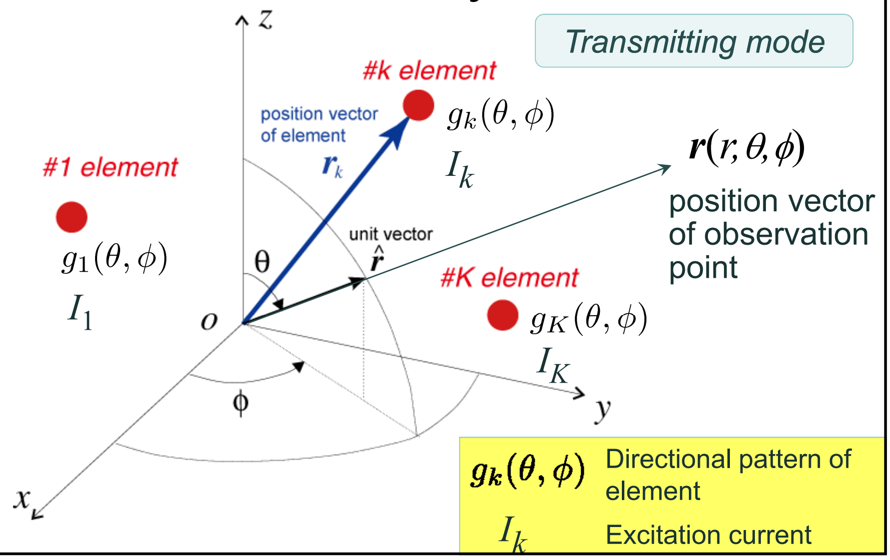
Why array antenna is used?

- SNR increased by in-phase combination of array-element signals
- High angular resolution with narrow mainlobe
- Electronic scan of mainlobe/null

General configuration of *K*-element array antenna



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Radiation field from antenna at origin (phase center)

Maxwell's equation

spherical wave

Focus on θ – component

$$\boldsymbol{E}(r,\theta,\phi) = \zeta_0 I_0 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r}\left[\hat{\boldsymbol{\theta}}\, g_\theta(\theta,\phi) + \hat{\boldsymbol{\phi}}\, g_\phi(\theta,\phi)\right]$$

No r-component

ζ_0: constant

I_0: excitation current

λ: wavelength

$g_\theta(\theta,\phi)$: directional function of θ-component

$g_\phi(\theta,\phi)$: directional function of ϕ-component

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Approximation of radiation field (e.g., θ-component)

#1 $$E_1(r,\theta,\phi) = \zeta_0 I_1 \frac{\exp\left(-j\frac{2\pi}{\lambda}r'_1\right)}{r'_1} g_1(\theta,\phi)$$

$$= \zeta_0 I_1 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} g_1(\theta,\phi)\exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_1^T\hat{\boldsymbol{r}}\right)$$

phase difference from origin

#2 $$E_2(r,\theta,\phi) = \zeta_0 I_2 \frac{\exp\left(-j\frac{2\pi}{\lambda}r'_2\right)}{r'_2} g_2(\theta,\phi)$$

$$= \zeta_0 I_2 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} g_2(\theta,\phi)\exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_2^T\hat{\boldsymbol{r}}\right)$$

phase difference from origin

.
.
.

I_k, $\boldsymbol{r}_k$, $g_k(\theta,\phi)$: excitation current, position vector and directional function of k-th element

r'_k: distance from k-th element to observation point

$\hat{\boldsymbol{r}}$: unit vector in (θ,ϕ) (look direction)

Approximation of radiation field

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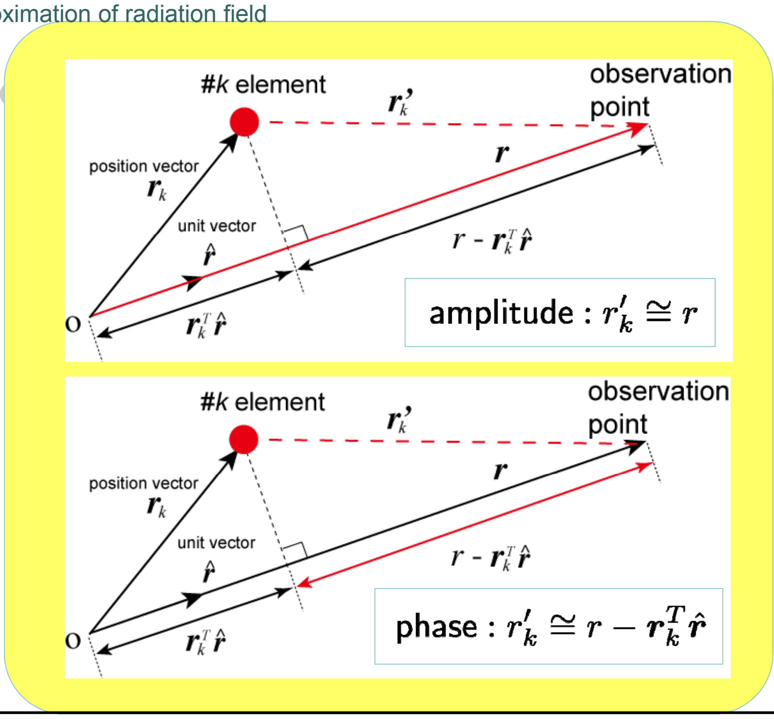

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Radiation field of array antenna (Combined electric field)

spherical wave — phase difference due to element position

$$E(r,\theta,\phi) = \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} \sum_{k=1}^{K} I_k g_k(\theta,\phi) \exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_k^T \hat{\boldsymbol{r}}\right)$$

$$= \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} E_0(\theta,\phi)$$

$$E_0(\theta,\phi) = \sum_{k=1}^{K} I_k g_k(\theta,\phi) \exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_k^T \hat{\boldsymbol{r}}\right)$$

Directional function of array antenna

I_k: excitation current of k-th element

$\boldsymbol{r}_k$: position vector of k-th element (x_k, y_k, z_k)

$\hat{\boldsymbol{r}}$: unit vector in (θ,ϕ) (look direction)

$g_k(\theta,\phi)$: directional function of k-th element

λ: wavelength, ζ_0: constant

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Radiation field of array antenna (Cont'd)

- If directional functions of all elements are identical, then

$$E(r,\theta,\phi) = \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} g(\theta,\phi) \sum_{k=1}^{K} I_k \exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_k^T \hat{\boldsymbol{r}}\right)$$

$$= \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda}r\right)}{r} E_0(\theta,\phi)$$

$$E_0(\theta,\phi) = g(\theta,\phi)A(\theta,\phi)$$

Principle of pattern multiplication

$$A(\theta,\phi) = \sum_{k=1}^{K} I_k \exp\left(j\frac{2\pi}{\lambda}\boldsymbol{r}_k^T \hat{\boldsymbol{r}}\right)$$ **: array factor**

$$g(\theta,\phi) = g_k(\theta,\phi)$$ **: element pattern**

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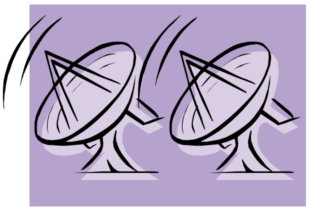
Directivity and its optimization



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Directivity of array antenna

definition

$$G(\theta_0,\phi_0) = \frac{4\pi r^2 |E(r,\theta_0,\phi_0)|^2 / Z_0}{\iint \frac{|E(r,\theta,\phi)|^2}{Z_0} r^2 \sin\theta d\theta d\phi}$$

$$= \frac{4\pi |E_0(\theta_0,\phi_0)|^2}{\iint |E_0(\theta,\phi)|^2 \sin\theta d\theta d\phi}$$

$$= \frac{4\pi |A(\theta_0,\phi_0)|^2}{\iint |A(\theta,\phi)|^2 f(\theta,\phi) \sin\theta d\theta d\phi}$$

Z_0 impedance

$|E|^2/Z_0$ power density

Identical elements

$E_0(\theta,\phi) = g(\theta,\phi)A(\theta,\phi)$

$$f(\theta,\phi) = \frac{|g(\theta,\phi)|^2}{|g(\theta_0,\phi_0)|^2}$$

Normalized power pattern of antenna element

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Directivity of array antenna (Cont'd)

Vector notation

$\boldsymbol{w} = [I_1, I_2, \cdots, I_K]^T$: excitation current vector

$$\boldsymbol{v}(\theta,\phi) = \left[\exp\left(-j\frac{2\pi}{\lambda}\boldsymbol{r}_1^T\hat{\boldsymbol{r}}\right), \cdots, \exp\left(-j\frac{2\pi}{\lambda}\boldsymbol{r}_K^T\hat{\boldsymbol{r}}\right)\right]^T$$

: array steering vector

$A(\theta,\phi) = \boldsymbol{v}^H(\theta,\phi)\boldsymbol{w}$ — Inner product of above two vectors

Directivity

$$G(\theta_0,\phi_0) = \frac{\boldsymbol{w}^H \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{B} \boldsymbol{w}}$$

Ratio of two Hermitian forms

$$\begin{cases} \boldsymbol{A} = \boldsymbol{v}(\theta_0,\phi_0)\boldsymbol{v}^H(\theta_0,\phi_0) \\ \boldsymbol{B} = \frac{1}{4\pi}\iint \boldsymbol{v}(\theta,\phi)\boldsymbol{v}^H(\theta,\phi) f(\theta,\phi)\sin\theta d\theta d\phi \end{cases}$$

Maximization of directivity of array antenna

$$G(\theta_0, \phi_0) = \frac{\boldsymbol{w}^H \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{B} \boldsymbol{w}}$$

Generalized Rayleigh quotient

What is $\boldsymbol{w}$ maximizing G?

$$\boldsymbol{A} = \boldsymbol{v}_0 \boldsymbol{v}_0^H \quad \boldsymbol{v}_0 = \boldsymbol{v}(\theta_0, \phi_0)$$

$$\boldsymbol{B} = \frac{1}{4\pi} \iint \boldsymbol{v}(\theta, \phi) \boldsymbol{v}^H(\theta, \phi) f(\theta, \phi) \sin\theta d\theta d\phi$$

Generalized eigenvalue problem:

$$\lambda = \frac{\boldsymbol{w}^H \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{B} \boldsymbol{w}} = G$$

[W,Lambda]=eig(A,B);

$$\boldsymbol{A}\boldsymbol{w} = \lambda \boldsymbol{B} \boldsymbol{w}$$

$\boldsymbol{w}_M$: Eigenvector corresponding to maximum eigenvalue λ_M

$$\boldsymbol{w}_M = \boldsymbol{B}^{-1} \boldsymbol{v}_0 \quad \lambda_M = G_{max} = \boldsymbol{v}_0^H \boldsymbol{B}^{-1} \boldsymbol{v}_0$$

Maximization of directivity of array antenna (Cont'd) *Example*

- Uniform linear array of isotropic elements with element spacing of d

$$\boldsymbol{B} = [b_{nm}] \quad (n, m = 1, \cdots, K)$$

$$b_{nm} = \frac{\sin\left\{\frac{2\pi}{\lambda}(n-m)d\right\}}{\frac{2\pi}{\lambda}(n-m)d}$$

- In addition, when $d = \lambda/2$,

$$b_{nm} = \delta_{nm} \Rightarrow \boldsymbol{B} = \boldsymbol{I}$$ (δ_{nm} : Kronecker delta)

$\boldsymbol{w}_M = \boldsymbol{v}_0$ — **Uniform excitation**

$G_{max} = \boldsymbol{v}_0^H \boldsymbol{v}_0 = K$ — **Number of elements**

Q factor & mainbeam radiation efficiency of array antenna

Q factor

$$Q = \frac{\boldsymbol{w}^H \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{B} \boldsymbol{w}}$$

$$\boldsymbol{B} = \frac{1}{4\pi} \iint \boldsymbol{v}(\theta,\phi)\boldsymbol{v}^H(\theta,\phi) f(\theta,\phi) \sin\theta d\theta d\phi$$

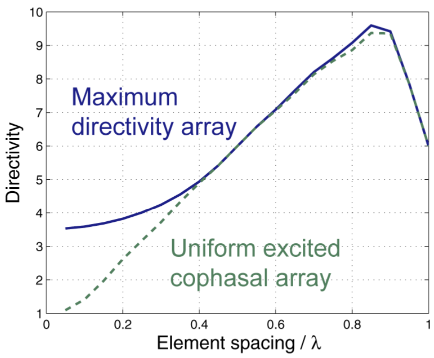
Mainbeam radiation efficiency

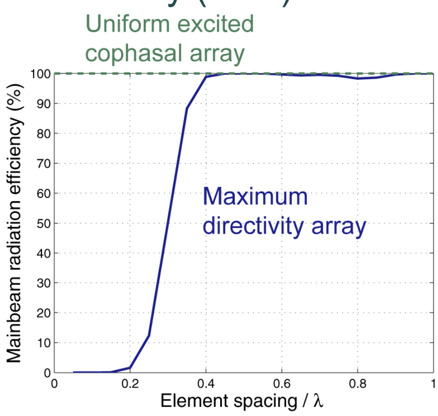
$$\eta(\theta_0,\phi_0) = \frac{1}{K}\frac{\boldsymbol{w}^H \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{w}} \qquad \boldsymbol{v}_0 = \arg\max_{\boldsymbol{w}} \eta(\theta_0,\phi_0)$$

$$\boldsymbol{A} = \boldsymbol{v}_0 \boldsymbol{v}_0^H \qquad \boldsymbol{v}_0 = \boldsymbol{v}(\theta_0,\phi_0)$$

Numerical Examples

6-element uniform linear array (ULA)

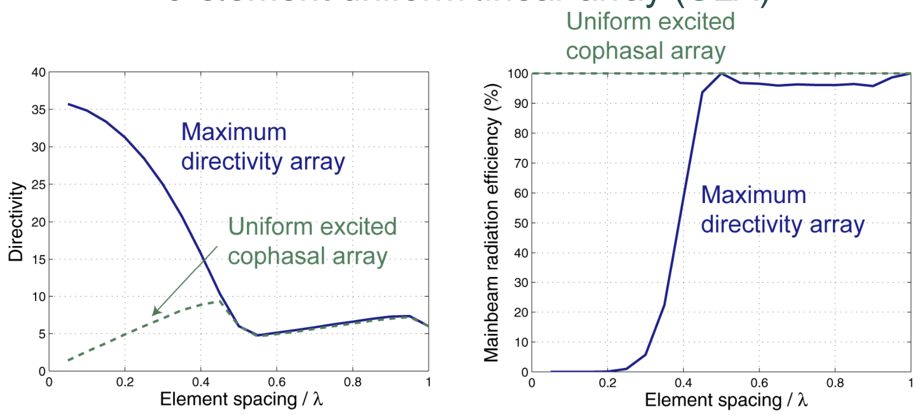




Broadside array with isotropic elements $(\theta_0 = 0\text{deg})$

Numerical Examples (Cont'd)

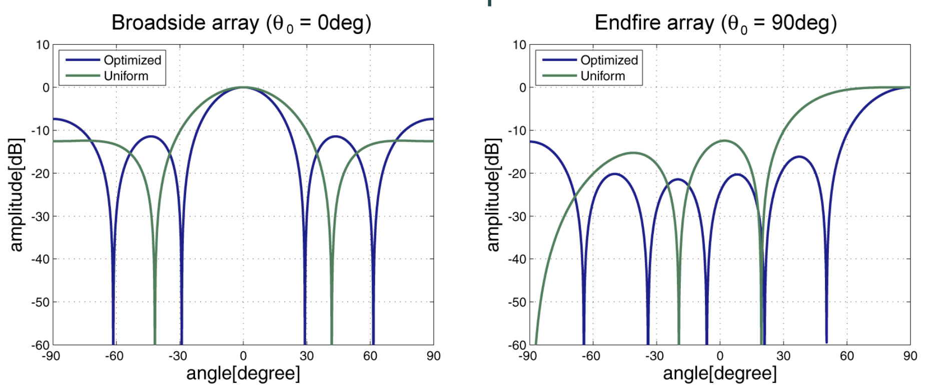
6-element uniform linear array (ULA)



Endfire array with isotropic elements $(\theta_0 = 90\text{deg})$

Numerical Examples (Cont'd)

6-element uniform linear array with $d = \lambda/4$
Directional patterns



$\boldsymbol{w}_\mathrm{M}$ =[1.11, -0.91, 0.80, 0.80, -0.91, 1.11] ^{T} $\boldsymbol{w}_\mathrm{M}$ =[-2.06+1.71j, 6.92-4.15j, -11.73+5.10j, 12.28 - 3.59j, -7.98 + 1.23j, 2.68] ^{T}

$\boldsymbol{v}_0$=[1, 1, 1, 1, 1, 1] ^{T} $\boldsymbol{v}_0$=[1, -j, -1, j, 1, -j] ^{T}

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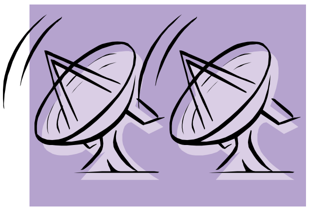
Adaptive array and its optimization



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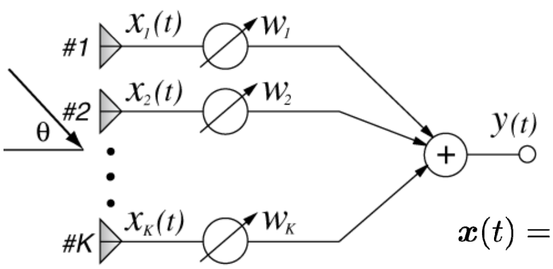
Adaptive array model

(receiving mode)



$x(t)$: input
w : weight
$y(t)$: output

K-element array antenna

Vector notation of inputs and weights

$$\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_K(t)]^T$$

$$\boldsymbol{w} = [w_1, w_2, \ldots, w_K]^T$$

$$y(t) = \boldsymbol{w}^H \boldsymbol{x}(t) = \boldsymbol{x}^T(t)\boldsymbol{w}^*$$

Output is expressed as inner product of two vectors.

Adaptive array model (Cont'd)

- Array output power:

$$P_{out} = \frac{1}{2}E[|y(t)|^2] = \frac{1}{2}\boldsymbol{w}^H \boldsymbol{R}_{xx}\boldsymbol{w}$$

Hermitian form

- Covariance matrix:

$$\boldsymbol{R}_{xx} = E[\boldsymbol{x}(t)\boldsymbol{x}^H(t)]$$

Hermitian matrix

Adaptive array model (Cont'd)

- Output SINR:

$$\text{SINR} = \frac{\text{Desired signal}}{\text{Interference} + \text{Internal noise}}$$

at output

Large value means good receiving performance → Maximization of SINR

*SINR: Signal-to-Interference-plus-Noise Ratio

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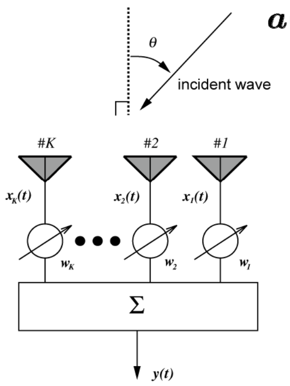
Adaptive array output (1)
(linear array, no interference)

Input vector: $\boldsymbol{x}(t) = \underbrace{s(t)\boldsymbol{a}(\theta_0)}_{S} + \underbrace{\boldsymbol{n}(t)}_{N}$

$$\boldsymbol{a}(\theta) = \left[g_1(\theta)\exp\left(-j\frac{2\pi}{\lambda}d_1\sin\theta\right) \cdots \right.$$
$$\left. g_K(\theta)\exp\left(-j\frac{2\pi}{\lambda}d_K\sin\theta\right) \right]^T$$



$s(t)$: signal amplitude at phase center

$\boldsymbol{a}(\theta)$: array response vector in θ

$\boldsymbol{n}(t)$: internal noise vector

$g_k(\theta)$: directional function of k-th element

θ_0 : DOA of signal, d_k : element position

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Adaptive array output (1) (Cont'd)
(linear array, no interference)

Array output:

$$y(t) = s(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_0) + \boldsymbol{w}^H\boldsymbol{n}(t)$$

$$\text{SINR} = \frac{E[|s(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_0)|^2]}{E[|\boldsymbol{w}^H\boldsymbol{n}(t)|^2]} = \frac{E[|s(t)|^2]|\boldsymbol{w}^H\boldsymbol{a}(\theta_0)|^2}{\boldsymbol{w}^H E[\boldsymbol{n}(t)\boldsymbol{n}^H(t)]\boldsymbol{w}} = \frac{P_s}{P_n}\frac{\boldsymbol{w}^H\boldsymbol{a}(\theta_0)\boldsymbol{a}^H(\theta_0)\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{w}}$$

Input SNR ($\frac{P_s}{P_n}$)

Rayleigh quotient ($\frac{\boldsymbol{w}^H\boldsymbol{a}(\theta_0)\boldsymbol{a}^H(\theta_0)\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{w}}$)

$P_s = E[|s(t)|^2]$ (signal power)

P_n: internal noise power

$E[\boldsymbol{n}(t)\boldsymbol{n}^H(t)] = P_n\boldsymbol{I}$

Equal to mainbeam radiation efficiency

Weight vector maximizing SINR: $\boldsymbol{w} = \boldsymbol{a}(\theta_0)$

Adaptive array output (2)

(linear array, L interferences)

Input vector:

$$\boldsymbol{x}(t) = \underbrace{s(t)\boldsymbol{a}(\theta_0)}_{S} + \underbrace{\sum_{i=1}^{L} u_i(t)\boldsymbol{a}(\theta_i)}_{I} + \underbrace{\boldsymbol{n}(t)}_{N}$$

$s(t)$: amplitude of desired signal

$u_i(t)$: amplitude of i-th interference

θ_0 : DOA of desired signal

θ_i : DOA of i-th interference

$\boldsymbol{a}(\theta)$: array response vector in θ

$\boldsymbol{n}(t)$: internal noise vector

Adaptive array output (2) (Cont'd)

(linear array, L interferences)

Array output:

$$y(t) = s(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_0) + \sum_{i=1}^{L} u_i(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_i) + \boldsymbol{w}^H\boldsymbol{n}(t)$$

$$\mathrm{SINR} = \frac{E[|s(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_0)|^2]}{\sum_{i=1}^{L} E[|u_i(t)\boldsymbol{w}^H\boldsymbol{a}(\theta_i)|^2] + E[|\boldsymbol{w}^H\boldsymbol{n}(t)|^2]}$$

$$= \frac{\boldsymbol{w}^H\boldsymbol{R}_{ss}\boldsymbol{w}}{\boldsymbol{w}^H\boldsymbol{R}_{nn}\boldsymbol{w}}$$ Generalized Rayleigh quotient

$$\boldsymbol{R}_{ss} = P_0\boldsymbol{a}(\theta_0)\boldsymbol{a}^H(\theta_0)$$ ($P_0 = P_S$)

$$\boldsymbol{R}_{nn} = \sum_{i=1}^{L} P_i\boldsymbol{a}(\theta_i)\boldsymbol{a}^H(\theta_i) + P_n\boldsymbol{I}$$

$P_0 = E[|s(t)|^2]$, $P_i = E[|u_i(t)|^2]$, P_n : internal noise power

Adaptive array output (2) (Cont'd)

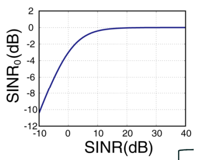
(linear array, *L interferences*)

$$\frac{\boldsymbol{w}^H \boldsymbol{R}_{ss} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}} = \frac{\mathrm{SINR}}{\mathrm{SINR}+1} \equiv \mathrm{SINR}_0$$

Quasi-Normalized SINR

Maximization of SINR

$$\equiv \text{Maximization of } \mathrm{SINR}_0 = \frac{\boldsymbol{w}^H \boldsymbol{R}_{ss} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}}$$



Generalized Rayleigh quotient

$$\boldsymbol{R}_{ss} = P_0 \boldsymbol{a}(\theta_0) \boldsymbol{a}^H(\theta_0)$$

$$\boldsymbol{R}_{xx} = P_0 \boldsymbol{a}(\theta_0) \boldsymbol{a}^H(\theta_0) + \sum_{i=1}^{L} P_i \boldsymbol{a}(\theta_i) \boldsymbol{a}^H(\theta_i) + P_n \boldsymbol{I}$$

Directivity and Output SINR

$$G(\theta_0, \phi_0) = \frac{\boldsymbol{w}^H \boldsymbol{A} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{B} \boldsymbol{w}}$$

Same form

$$\boldsymbol{A} = \boldsymbol{v}_0 \boldsymbol{v}_0^H \qquad \boldsymbol{v}_0 = \boldsymbol{v}(\theta_0, \phi_0)$$

$$\boldsymbol{B} = \frac{1}{4\pi} \iint \boldsymbol{v}(\theta, \phi) \boldsymbol{v}^H(\theta, \phi) f(\theta, \phi) \sin\theta d\theta d\phi$$

$$\mathrm{SINR}_0 = \frac{\boldsymbol{w}^H \boldsymbol{R}_{ss} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}}$$

$$\boldsymbol{R}_{ss} = P_0 \boldsymbol{v}_0 \boldsymbol{v}_0^H \qquad \boldsymbol{v}_0 = \boldsymbol{v}(\theta_0, \phi_0)$$

Desired signal

$$\boldsymbol{R}_{xx} = \iint \boldsymbol{v}(\theta, \phi) \boldsymbol{v}^H(\theta, \phi) f(\theta, \phi) p(\theta, \phi) \sin\theta d\theta d\phi + P_n \boldsymbol{I}$$

$$p(\theta, \phi) = \sum_{i=0}^{L} P_i \delta(\theta - \theta_i) \delta(\phi - \phi_i)$$

Angular distribution of incident waves

Typical criteria of adaptive array

- Maximum Signal-to-Noise Ration: MSN
- Minimum Mean Square Error: MMSE

MSN adaptive array

- This adaptive array controls weights to maximize the output SNR(SINR).
- A priori knowledge: DOA of desired signal

Cost function:

$$\mathrm{SINR_o} = \frac{\boldsymbol{w}^H \boldsymbol{R}_{ss} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}}$$ *maximized*

$$\boldsymbol{R}_{ss} = P_0 \boldsymbol{v}_0 \boldsymbol{v}_0^H \quad \boldsymbol{v}_0 = \boldsymbol{v}(\theta_0, \phi_0) \ \text{(known)}$$

$$\boldsymbol{R}_{xx} = \iint \boldsymbol{v}(\theta,\phi)\boldsymbol{v}^H(\theta,\phi) f(\theta,\phi) p(\theta,\phi) \sin\theta d\theta d\phi + P_n \boldsymbol{I}$$

$$p(\theta,\phi) = \sum_{i=0}^{L} P_i \delta(\theta - \theta_i)\delta(\phi - \phi_i)$$

Angular distribution of incident waves

MSN adaptive array (Cont'd)

- Optimum weight vector:

$$\boldsymbol{w}_{opt} = \boldsymbol{R}_{xx}^{-1}\boldsymbol{v}_0$$

$\boldsymbol{v}_0$: steering vector

MSN adaptive array (Cont'd)

- Optimum weight vector $\boldsymbol{w}'_{opt}$ maximizing

SINR $= \dfrac{\boldsymbol{w}^H \boldsymbol{R}_{ss} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{nn} \boldsymbol{w}}$

$$\boldsymbol{w}'_{opt} = \boldsymbol{R}_{nn}^{-1}\boldsymbol{v}_0 = \gamma \boldsymbol{R}_{xx}^{-1}\boldsymbol{v}_0 = \gamma \boldsymbol{w}_{opt}$$

(γ: constant)

$$\boldsymbol{R}_{xx} = P_0 \boldsymbol{v}_0 \boldsymbol{v}_0^H + \boldsymbol{R}_{nn}$$

$$\boldsymbol{R}_{xx}^{-1} = \boldsymbol{R}_{nn}^{-1} - \frac{\boldsymbol{R}_{nn}^{-1} P_0 \boldsymbol{v}_0 \boldsymbol{v}_0^H \boldsymbol{R}_{nn}^{-1}}{1 + P_0 \boldsymbol{v}_0^H \boldsymbol{R}_{nn}^{-1} \boldsymbol{v}_0}$$

$$\boldsymbol{R}_{xx}^{-1}\boldsymbol{v}_0 = \boldsymbol{R}_{nn}^{-1}\boldsymbol{v}_0 - \frac{\boldsymbol{R}_{nn}^{-1} P_0 \boldsymbol{v}_0 \boldsymbol{v}_0^H \boldsymbol{R}_{nn}^{-1} \boldsymbol{v}_0}{1 + P_0 \boldsymbol{v}_0^H \boldsymbol{R}_{nn}^{-1} \boldsymbol{v}_0} = \boldsymbol{R}_{nn}^{-1}\boldsymbol{v}_0 \underbrace{\frac{1}{1 + P_0 \boldsymbol{v}_0^H \boldsymbol{R}_{nn}^{-1} \boldsymbol{v}_0}}_{\gamma}$$

MMSE adaptive array

- Minimizing error $e(t)$ which is a difference between reference signal $r(t)$ and array output $y(t)$:

$$e(t) = r(t) - y(t) = r(t) - \boldsymbol{w}^H \boldsymbol{x}(t)$$

Cost function: $E\left[|e(t)|^2\right] = E\left[|r(t) - y(t)|^2\right]$

$$= E\left[|r(t) - \boldsymbol{w}^H \boldsymbol{x}(t)|^2\right]$$

$$= E\left[|r(t)|^2\right] - \boldsymbol{w}^T \boldsymbol{r}_{xr}^* - \boldsymbol{w}^H \boldsymbol{r}_{xr} + \boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}$$

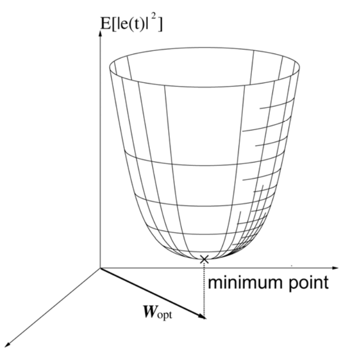
minimized

$$\boldsymbol{r}_{xr} = E[\boldsymbol{x}(t) r^*(t)] \text{(correlation vector)}$$

MMSE adaptive array (Cont'd)

- Optimum weight vector

$$\boldsymbol{w}_{opt} = \boldsymbol{R}_{xx}^{-1} \boldsymbol{r}_{xr}$$



In the case of one desired signal incident

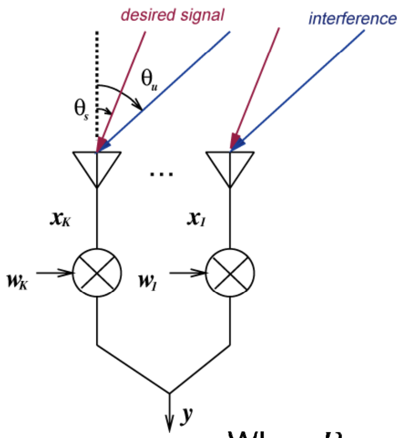
$$\boldsymbol{r}_{xr} = \alpha \boldsymbol{v}_0$$

$$(\alpha: \text{ constant})$$

This weight is equal to that of MSN adaptive array.

Maximizing output SINR

Example of 6-element adaptive array (*K* = 6)

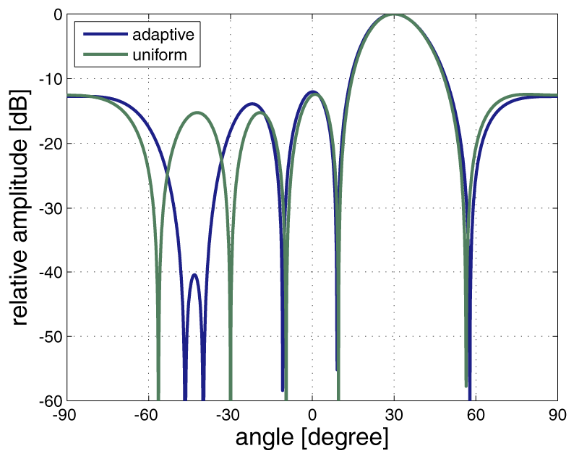


- desired signal: $s(t)$, P_s, θ_s
- interference: $u(t)$, P_u, θ_u
- antenna elements: isotropic
- element spacing: $\lambda/2$
- MMSE: $r(t) = s(t)$
- MSN: θ_s is completely known

When $P_s = P_u = 1$, $P_n = 0.01$, $\theta_s = 30°$, $\theta_u = -40°$ (SNR=20dB, SIR=0dB)

Example of 6-element adaptive array (*K* = 6) (Cont'd)

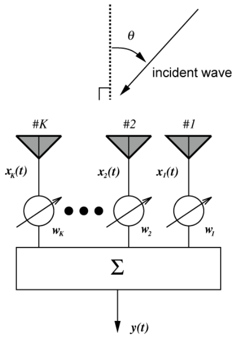
Directional patterns



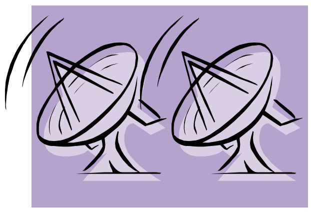
Adaptive
(MMSE & MSN)
SINR = 27.66dB
SINR_0 = 0.9983

Uniform
SINR = 15.31dB
SINR_0 = 0.9714

DOA estimation and array optimization



Linear array



Beam-scan scheme

- Beamformer

Maximization of directivity in θ

$$\max_{\boldsymbol{w}} \left(\frac{1}{2} \boldsymbol{w}^H \boldsymbol{a}(\theta) \boldsymbol{a}^H(\theta) \boldsymbol{w} \right) \quad \text{subject to } \boldsymbol{w}^H \boldsymbol{w} = 1$$

$$P_{BF}(\theta) = \boldsymbol{w}_{opt}^H \boldsymbol{R}_{xx} \boldsymbol{w}_{opt} = \frac{\boldsymbol{a}^H(\theta) \boldsymbol{R}_{xx} \boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta) \boldsymbol{a}(\theta)}$$

Angular spectrum

- Capon Beamformer

Constrained minimization

$$\min_{\boldsymbol{w}} \left(P_{out} = \frac{1}{2} \boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w} \right) \quad \text{subject to } \boldsymbol{w}^H \boldsymbol{a}(\theta) = 1$$

$$P_{CP}(\theta) = \boldsymbol{w}_{opt}^H \boldsymbol{R}_{xx} \boldsymbol{w}_{opt} = \frac{1}{\boldsymbol{a}^H(\theta) \boldsymbol{R}_{xx}^{-1} \boldsymbol{a}(\theta)}$$

Angular spectrum

Beam-scan scheme
(Rayleigh quotient expression)

- Beamformer

Mainbeam radiation efficiency

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^H \boldsymbol{a}(\theta)\boldsymbol{a}^H(\theta)\boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{w}} \Rightarrow \boldsymbol{w}_{opt} = \alpha\, \boldsymbol{a}(\theta)$$

α: scalar

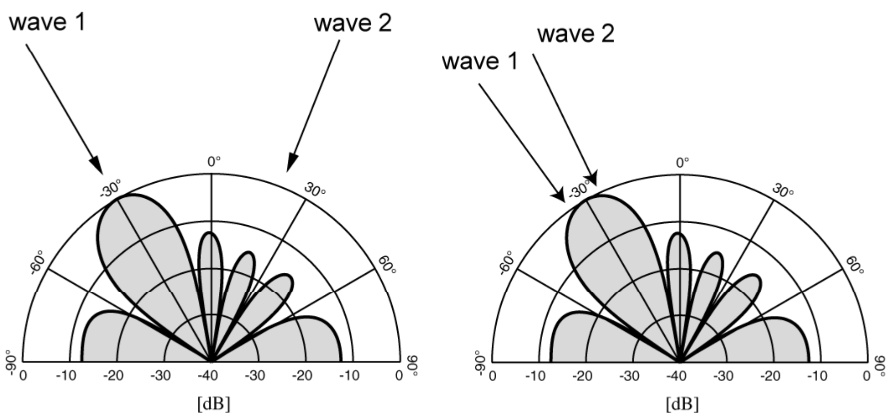
- Capon Beamformer

SINR

$$\max_{\boldsymbol{w}} \frac{\boldsymbol{w}^H \boldsymbol{a}(\theta)\boldsymbol{a}^H(\theta)\boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}} \Rightarrow \boldsymbol{w}_{opt} = \gamma \boldsymbol{R}_{xx}^{-1} \boldsymbol{a}(\theta)$$

γ: scalar

Problems of Beamformer



Wave 2 is received by sidelobe.

Closely spaced waves cannot be separated.

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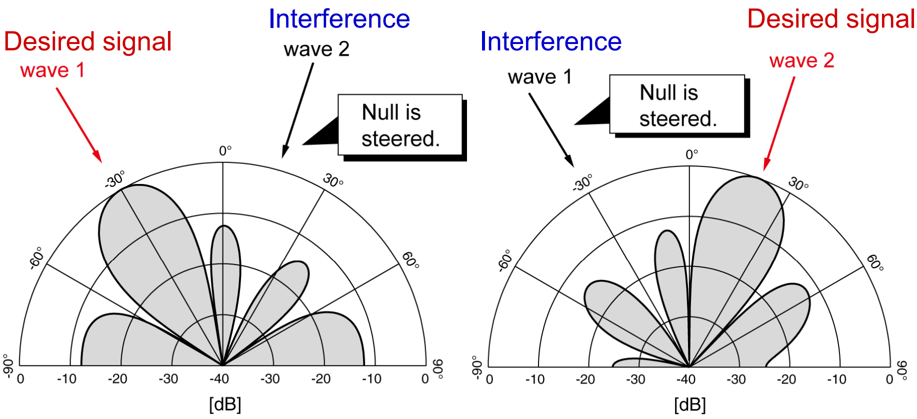
Advantage of Capon Beamformer



Examples of directional patterns by optimum weights

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How angular resolution is enhanced furthermore?

Nulls are utilized.

Null-scan scheme

Nulls are adaptively steered in the respective directions of incident waves.

Angular spectrum of null-scan scheme:

$$P(\theta) = \frac{1}{\left| \boldsymbol{w}_{opt}^{H} \boldsymbol{a}(\theta) \right|^2}$$

Inverse of power pattern

Null-scan scheme (Constrained minimization)

- linearly constrained minimization (Linear prediction)

$$\min_{\boldsymbol{w}} \left(P_{out} = \frac{1}{2} \boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w} \right) \text{ subject to } \boldsymbol{w}^H \boldsymbol{u} = 1$$

$$\boldsymbol{w}_{opt} = \boldsymbol{R}_{xx}^{-1} \boldsymbol{u} \qquad \boldsymbol{u} = [1, 0, \cdots, 0]^T$$

- Quadratically constrained minimization

$$\min_{\boldsymbol{w}} \left(P_{out} = \frac{1}{2} \boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w} \right) \text{ subject to } \boldsymbol{w}^H \boldsymbol{w} = 1$$

$\boldsymbol{w}_{opt}$: eigenvectors corresponding to minimum eigenvalues of $\boldsymbol{R}_{xx}$

⇒ Pisarenko's method, Min-Norm method, MUSIC

Null-scan scheme (Rayleigh quotient expression)

- linearly constrained minimization (Linear prediction)

$$\min_{\boldsymbol{w}} \frac{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{u} \boldsymbol{u}^H \boldsymbol{w}} \Rightarrow \boldsymbol{w}_{opt} = \boldsymbol{R}_{xx}^{-1} \boldsymbol{u}$$

$$\boldsymbol{u} = [1, 0, \cdots, 0]^T$$

- Quadratically constrained minimization

$$\min_{\boldsymbol{w}} \frac{\boldsymbol{w}^H \boldsymbol{R}_{xx} \boldsymbol{w}}{\boldsymbol{w}^H \boldsymbol{w}} \Rightarrow \boldsymbol{w}_{opt} = \boldsymbol{e}_{min}$$

(Pisarenko's method)

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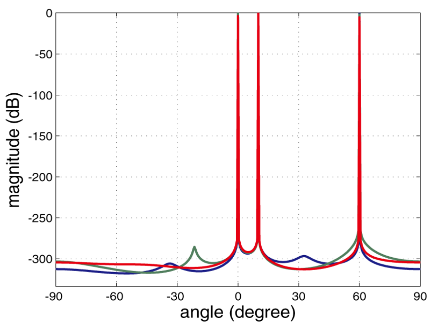
Eigenvectors of minimum eigenvalues of covariance matrix

eigenvalues when K-element array receives $L(L < K)$ waves

Eigenvalues: $\lambda_1 \geq \cdots \geq \lambda_L > \lambda_{L+1} = \cdots = \lambda_K = \sigma^2$

Eigenvectors: $\boldsymbol{e}_1 \quad \cdots \quad \boldsymbol{e}_L \quad \boldsymbol{e}_{L+1} \quad \cdots \quad \boldsymbol{e}_K$ Noise power



Pisarenko's method

Example of 6-element ULA with element spacing of a half wavelength (*L*=3)

3 eigenvectos used individually

Some smoothing required

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Min-Norm method and MUSIC

Min-Norm method

Linear combination of eigenvectors

$$\boldsymbol{w} = \alpha_{L+1}\boldsymbol{e}_{L+1} + \cdots + \alpha_K \boldsymbol{e}_K = \boldsymbol{E}_N \boldsymbol{\alpha}$$

$$\min_{\boldsymbol{w}}(\boldsymbol{w}^H \boldsymbol{w}) \text{ subject to } (\boldsymbol{E}_S^H \boldsymbol{w} = 0 \text{ and } \boldsymbol{w}^H \boldsymbol{u} = 1)$$

$$\boldsymbol{E}_S = [\boldsymbol{e}_1, \cdots, \boldsymbol{e}_L], \ \boldsymbol{E}_N = [\boldsymbol{e}_{L+1}, \cdots, \boldsymbol{e}_K], \ \boldsymbol{u} = [1, 0, \cdots, 0]^T$$

Signal subspace Noise subspace $\boldsymbol{\alpha} = [\alpha_{L+1}, \cdots, \alpha_K]^T$

$$P_{MN}(\theta) = \frac{1}{\left|\boldsymbol{w}_{opt}^H \boldsymbol{a}(\theta)\right|^2} \times \boldsymbol{a}^H(\theta)\boldsymbol{a}(\theta)$$

MUSIC

$$P_{MUSIC}(\theta) = \frac{\boldsymbol{a}^H(\theta)\boldsymbol{a}(\theta)}{|\boldsymbol{e}_{L+1}^H \boldsymbol{a}(\theta)|^2 + \cdots + |\boldsymbol{e}_K^H \boldsymbol{a}(\theta)|^2}$$

$$= \frac{\boldsymbol{a}^H(\theta)\boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\boldsymbol{E}_N \boldsymbol{E}_N^H \boldsymbol{a}(\theta)}$$

Combination of power patterns

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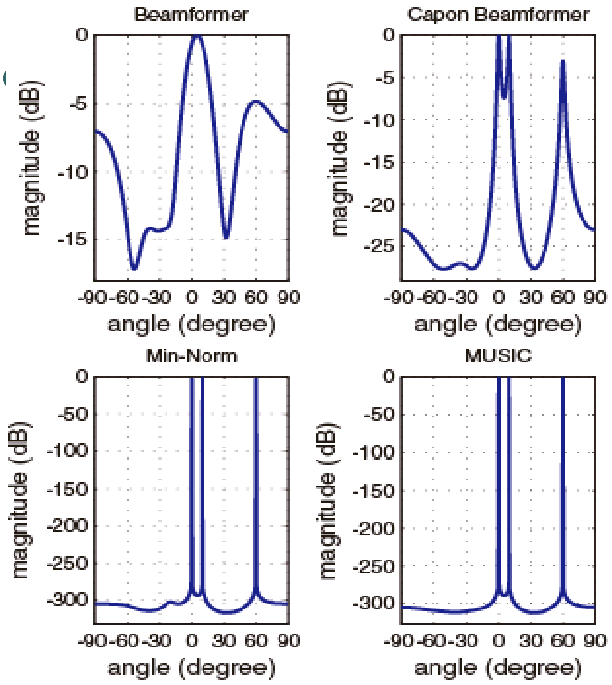

5 estimators

6-element ULA

3 waves
0˚, 1
10˚, 1
60˚, 0.5

Noise power
0.01

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Summary and conclusions

- Optimization of array antennas using eigenvalues and eigenvectors
  - Directivity
  - Adaptive arrays
  - DOA estimation
- Rayleigh quotient expression
  - Link to eigenvalue problem
  - Optimization of array antennas based on Rayleigh quotient expression

Unified array antenna theory including adaptive arrays and DOA estimation