Eigenvalues and Eigenvectors in Array Antennas

Optimization of Array Antennas for High Performance

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Self-introduction

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Research interests:
• adaptive and signal processing array
• multipath propagation analysis
• mobile and indoor wireless communication
• wireless power transmission

Hobby:
• MUSIC (karaoke, etc.)
On this short course

- Spatial signal processing technology using array antenna has been one of the important approaches for improving the performance of communications and radars.
- This short course expresses the optimization of the array antenna for its high performance in various applications.

On this short course (cont’d)

- Significance of eigenvalues and eigenvectors of various matrices used in the array antenna is explained.
- Key words:
  - Gain of the array antenna
  - Optimum weights of the adaptive array
  - Array weights for direction-of-arrival (DOA) estimation (e.g. MUSIC)

Applications of radio waves
Communications, Broadcasting, and Sensors

Antennas are important components.

Radio Propagation in Mobile Communications

- No line of sight
- Reflection, diffraction and scattering (multipath)
- Heavy fading
Example of antenna pattern for multipath waves (1)

Only one dominant wave is received.

Example of antenna pattern for multipath waves (2)

All multipath waves are combined appropriately.
Radio Propagation in Cellular Communication Systems

- Effective utilization of frequency resources
- Co-channel interference (CCI)

Techniques expected in antenna systems

- Fading recovery
- Suppressing or Canceling CCI

Control of directional pattern using array antenna:

Adaptive array signal processing (pattern optimization)
Why array antenna is used?

- **SNR increased** by in-phase combination of array-element signals
- **High angular resolution** with narrow mainlobe
- **Electronic scan** of mainlobe/null

General configuration of $K$-element array antenna

- Transmitting mode
- Directional pattern of element
- Excitation current
Radiation field from antenna at origin (phase center)

Maxwell's equation

\[ E(r, \theta, \phi) = \zeta_0 I_0 \exp \left( -\frac{j 2\pi}{\lambda} r \right) \left[ \hat{\theta} g_\theta(\theta, \phi) + \hat{\phi} g_\phi(\theta, \phi) \right] \]

\( \zeta_0 \): constant
\( I_0 \): excitation current
\( \lambda \): wavelength
\( g_\theta(\theta, \phi) \): directional function of \( \theta \)-component
\( g_\phi(\theta, \phi) \): directional function of \( \phi \)-component

Focus on \( \theta \)-component

No \( r \)-component

Approximation of radiation field (e.g., \( \theta \)-component)

\#1

\[ E_1(r, \theta, \phi) = \zeta_0 I_1 \exp \left( -\frac{j 2\pi}{\lambda} r' \right) g_1(\theta, \phi) \]

\[ E_2(r, \theta, \phi) = \zeta_0 I_2 \exp \left( -\frac{j 2\pi}{\lambda} r \right) g_2(\theta, \phi) \]

\( r', r_1, r_2, r_1^\prime \): distance from \( k \)-th element to observation point
\( \hat{r} \): unit vector in \((\theta, \phi)\) (look direction)

\( I_k, r_k, g_k(\theta, \phi) \): excitation current, position vector and directional function of \( k \)-th element
Approximation of radiation field

Radiation field of array antenna
(Combined electric field)

\[ E(r, \theta, \phi) = \zeta_0 \sum_{k=1}^{K} I_k g_k(\theta, \phi) \exp \left( \frac{2\pi}{\lambda} r_k^T \hat{r} \right) \]

\[ E_r(\theta, \phi) = \sum_{k=1}^{K} I_k g_k(\theta, \phi) \exp \left( \frac{j}{\lambda} r_k^T \hat{r} \right) \]  

\( I_k \): excitation current of \( k \)-th element  
\( r_k \): position vector of \( k \)-th element \((x_k, y_k, z_k)\)  
\( \hat{r} \): unit vector in \((\theta, \phi)\) (look direction)  
\( g_k(\theta, \phi) \): directional function of \( k \)-th element  
\( \lambda \): wavelength, \( \zeta_0 \): constant
Radiation field of array antenna (Cont’d)

- If directional functions of all elements are identical, then
  \[ E(r, \theta, \phi) = \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda} r\right)}{r} g(\theta, \phi) \sum_{k=1}^{K} I_k \exp\left(j \frac{2\pi}{\lambda} r_k \hat{r}\right) \]
  \[ = \zeta_0 \frac{\exp\left(-j\frac{2\pi}{\lambda} r\right)}{r} E_o(\theta, \phi) \]
  \[ E_o(\theta, \phi) = g(\theta, \phi) A(\theta, \phi) \]
  \[ A(\theta, \phi) = \sum_{k=1}^{K} I_k \exp\left(j \frac{2\pi}{\lambda} r_k \hat{r}\right) \]
  \[ g(\theta, \phi) = g_k(\theta, \phi) \]

Principle of pattern multiplication

Directivity and its optimization
Directivity of array antenna

\[ G(\theta_0, \phi_0) = \left( \frac{4\pi r^2}{\int \frac{|E(r, \theta, \phi)|^2}{Z_0} r^2 \sin \theta d\theta d\phi} \right) \frac{1}{Z_0} \]

\[ = \left( \frac{4\pi |E_0(\theta_0, \phi_0)|^2}{\int |E_0(\theta, \phi)|^2 \sin \theta d\theta d\phi} \right) \frac{1}{Z_0} \]

\[ f(\theta, \phi) = \frac{|g(\theta, \phi)|^2}{g(\theta_0, \phi_0)^2} \text{ Normalized power pattern of antenna element} \]

Directivity of array antenna (Cont’d)

\[ w = [I_1, I_2, \ldots, I_K]^T \] : excitation current vector

\[ \nu(\theta, \phi) = \left[ \exp \left( -j \frac{2\pi}{\lambda} r_1^T \hat{r} \right), \ldots, \exp \left( -j \frac{2\pi}{\lambda} r_K^T \hat{r} \right) \right]^T \] : array steering vector

\[ A(\theta, \phi) = \nu^H(\theta, \phi) w \]

\[ G(\theta_0, \phi_0) = \frac{w^H A w}{w^H B w} \]

\[ A = \nu(\theta_0, \phi_0) \nu^H(\theta_0, \phi_0) \]

\[ B = \frac{1}{4\pi} \int \nu(\theta, \phi) \nu^H(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi \]
Maximization of directivity of array antenna

\[ G(\theta_0, \phi_0) = \frac{w^H A w}{w^H B w} \]

What is \( w \) maximizing \( G \)?

\[ A = v_0 v_0^H \quad v_0 = v(\theta_0, \phi_0) \]
\[ B = \frac{1}{4\pi} \int \int v(\theta, \phi) v^H(\theta, \phi) f(\theta, \phi) \sin \theta d\theta d\phi \]

Generalized eigenvalue problem:
\[ A w = \lambda B w \]
\[ \lambda = \frac{w^H A w}{w^H B w} = G \]
\[ w_M : \text{Eigenvector corresponding to maximum eigenvalue } \lambda_M \]

\[ w_M = B^{-1} v_0 \quad \lambda_M = G_{\text{max}} = v_0^H B^{-1} v_0 \]

Maximization of directivity of array antenna (Cont’d)

Example

- Uniform linear array of isotropic elements with element spacing of \( d \)

\[ B = [b_{nm}] \quad (n, m = 1, \cdots, K) \]
\[ b_{nm} = \delta_{nm} \frac{\sin \left\{ \frac{2\pi}{\lambda} (n - m)d \right\}}{\frac{2\pi}{\lambda} (n - m)d} \]

- In addition, when \( d = \lambda/2 \),

\[ b_{nm} = \delta_{nm} \Rightarrow B = I \quad (\delta_{nm}: \text{Kronecker delta}) \]

\[ w_M = v_0 \quad G_{\text{max}} = v_0^H v_0 = K \]

Uniform excitation  Number of elements
Q factor & mainbeam radiation efficiency of array antenna

Q factor

\[ Q = \frac{w^H w}{w^H B w} \]

\[ B = \frac{1}{4\pi} \int \int v(\theta, \phi)v^H(\theta, \phi)f(\theta, \phi)\sin\theta d\theta d\phi \]

Mainbeam radiation efficiency

\[ \eta(\theta_0, \phi_0) = \frac{1}{K} \frac{w^H A w}{w^H w} \]

\[ v_0 = \arg\max_w \eta(\theta_0, \phi_0) \]

\[ A = v_0 v_0^H \]

\[ v_0 = v(\theta_0, \phi_0) \]

Numerical Examples

6-element uniform linear array (ULA)

Uniform excited cophasal array

Maximum directivity array

Broadside array with isotropic elements \((\theta_0 = 0\text{deg})\)
Numerical Examples (Cont’d)

6-element uniform linear array (ULA)

Endfire array with isotropic elements \( (\theta_0 = 90\text{deg}) \)

Numerical Examples (Cont’d)

6-element uniform linear array with \( d = \lambda/4 \)

Directional patterns

\[ \mathbf{w}_M = [1.11, -0.91, 0.80, 0.80, -0.91, 1.11]^T, \quad \mathbf{w}_H = [-2.06+1.71j, 6.92-4.15j, -11.73+5.10j, 12.28 - 3.59j, -7.98 + 1.23j, 2.68]^T \]

\[ \mathbf{v}_0 = [1, -j, -1, j, 1, -j]^T \]

\[ \mathbf{v}_0 = [1, -j, -1, j, 1, -j]^T \]
Adaptive array and its optimization

**Adaptive array model**
(receiving mode)

Vector notation of inputs and weights

\[
x(t) = [x_1(t), x_2(t), \ldots, x_K(t)]^T
\]

\[
w = [w_1, w_2, \ldots, w_K]^T
\]

Output is expressed as inner product of two vectors.

\[
y(t) = w^H x(t) = x^T(t) w^*
\]
Adaptive array model (Cont’d)

- **Array output power:**
  \[ P_{out} = \frac{1}{2} E[|y(t)|^2] = \frac{1}{2} w^H R_{xx} w \]
  Hermitian form

- **Covariance matrix:**
  \[ R_{xx} = E[x(t)x^H(t)] \]
  Hermitian matrix

Adaptive array model (Cont’d)

- **Output SINR:**
  \[ \text{SINR} = \frac{\text{Desired signal}}{\text{Interference + Internal noise at output}} \]
  Large value means good receiving performance
  
  *SINR: Signal-to-Interference-plus-Noise Ratio*
  
  **Maximization of SINR**
Adaptive array output (1) (linear array, no interference)

Input vector: \( \mathbf{x}(t) = s(t)\mathbf{a}(\theta) + \mathbf{n}(t) \)

\[ \mathbf{a}(\theta) = \begin{bmatrix} g_1(\theta) \exp \left( -j \frac{2\pi}{\lambda} d_1 \sin \theta \right) \\ \vdots \\ g_K(\theta) \exp \left( -j \frac{2\pi}{\lambda} d_K \sin \theta \right) \end{bmatrix}^T \]

\( s(t) \): signal amplitude at phase center
\( \mathbf{a}(\theta) \): array response vector in \( \theta \)
\( \mathbf{n}(t) \): internal noise vector
\( g_k(\theta) \): directional function of \( k \)-th element
\( \theta_s \): DOA of signal, \( d_k \): element position

Array output:
\[ y(t) = s(t)w^H \mathbf{a}(\theta_0) + w^H \mathbf{n}(t) \]

\[ \text{SINR} = \frac{E[|s(t)w^H \mathbf{a}(\theta_0)|^2]}{E[|w^H \mathbf{n}(t)|^2]} - \frac{E[|s(t)|^2|w^H \mathbf{a}(\theta_0)|^2]}{w^H E[\mathbf{n}(t)\mathbf{n}^H(t)]w} \]

\[ P_s = E[|s(t)|^2] \text{ (signal power)} \]
\[ P_n : \text{internal noise power} \]
\[ E[\mathbf{n}(t)\mathbf{n}^H(t)] = P_n I \]

Weight vector maximizing SINR: \( w = \mathbf{a}(\theta_0) \)
Adaptive array output (2)
(linear array, $L$ interferences)

Input vector:

$$x(t) = s(t)\mathbf{a}(\theta_0) + \sum_{i=1}^{L} u_i(t)\mathbf{a}(\theta_i) + \mathbf{n}(t)$$

- $s(t)$: amplitude of desired signal
- $u_i(t)$: amplitude of $i$-th interference
- $\theta_0$: DOA of desired signal
- $\theta_i$: DOA of $i$-th interference
- $\mathbf{a}(\theta)$: array response vector in $\theta$
- $\mathbf{n}(t)$: internal noise vector

Adaptive array output (2) (Cont’d)
(linear array, $L$ interferences)

Array output:

$$y(t) = s(t)\mathbf{w}^H\mathbf{a}(\theta_0) + \sum_{i=1}^{L} u_i(t)\mathbf{w}^H\mathbf{a}(\theta_i) + \mathbf{w}^H\mathbf{n}(t)$$

$$\text{SINR} = \frac{E[|s(t)\mathbf{w}^H\mathbf{a}(\theta_0)|^2]}{\sum_{i=1}^{L} E[|u_i(t)\mathbf{w}^H\mathbf{a}(\theta_i)|^2] + E[|\mathbf{w}^H\mathbf{n}(t)|^2]}$$

$$= \frac{\mathbf{w}^H\mathbf{R}_{ss}\mathbf{w}}{\mathbf{w}^H\mathbf{R}_{nn}\mathbf{w}}$$

Generalized Rayleigh quotient

$$R_{ss} = P_0\mathbf{a}(\theta_0)\mathbf{a}^H(\theta_0)$$

$$P_0 = P_S$$

$$R_{nn} = \sum_{i=1}^{L} P_i\mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i) + P_n\mathbf{I}$$

$$P_0 = E[|s(t)|^2], \quad P_i = E[|u_i(t)|^2], \quad P_n : \text{internal noise power}$$
Adaptive array output (2) (Cont’d)
(linear array, \(L\) interferences)

\[
\frac{w^H R_{ss}w}{w^H R_{xx}w} = \frac{\text{SINR}}{\text{SINR} + 1} \equiv \text{SINR}_0
\]

Quasi-Normalized SINR

Maximization of SINR

\[
\equiv \text{Maximization of } \text{SINR}_0 = \frac{w^H R_{ss}w}{w^H R_{xx}w}
\]

Generalized Rayleigh quotient

\[
R_{ss} = P_0a(\theta_0)a^H(\theta_0)
\]

\[
R_{xx} = P_0a(\theta_0)a^H(\theta_0) + \sum_{i=1}^{L} P_i a(\theta_i)a^H(\theta_i) + P_nI
\]

Directivity and Output SINR

\[G(\theta_0, \phi_0) = \frac{w^H Aw}{w^H Bw}\]

\[A = v_0v_0^H, \quad v_0 = v(\theta_0, \phi_0)\]

\[B = \frac{1}{4\pi} \int \int v(\theta, \phi)v^H(\theta, \phi)f(\theta, \phi)\sin \theta d\theta d\phi\]

\[\text{SINR}_0 = \frac{w^H R_{ss}w}{w^H R_{xx}w}\]

\[R_{ss} = P_0v_0v_0^H, \quad v_0 = v(\theta_0, \phi_0)\]

Desired signal

\[R_{xx} = \int \int v(\theta, \phi)v^H(\theta, \phi)f(\theta, \phi)p(\theta, \phi)\sin \theta d\theta d\phi + P_nI\]

\[p(\theta, \phi) = \sum_{i=0}^{L} P_i \delta(\theta - \theta_i)\delta(\phi - \phi_i)\]

Angular distribution of incident waves
Typical criteria of adaptive array

- **Maximum Signal-to-Noise Ratio**: $MSN$
- **Minimum Mean Square Error**: $MMSE$

MSN adaptive array

- This adaptive array controls weights to maximize the output SNR(SINR).
- A priori knowledge: DOA of desired signal

Cost function:

$$\text{SINR}_0 = \frac{w^H R_{ss} w}{w^H R_{xx} w}$$

Maximized

$$R_{ss} = P_0 v_0 v_0^H, \quad v_0 = v(\theta_0, \phi_0) \text{ (known)}$$

$$R_{xx} = \int \int v(\theta, \phi) v^H(\theta, \phi) f(\theta, \phi) p(\theta, \phi) \sin \theta d\theta d\phi + P_n I$$

$$p(\theta, \phi) = \sum_{i=0}^{L} P_i \delta(\theta - \theta_i) \delta(\phi - \phi_i)$$

Angular distribution of incident waves
Optimum weight vector:

\[ w_{opt} = R_{xx}^{-1}v_0 \]

\[ v_0: \text{steering vector} \]
MMSE adaptive array

- Minimizing error $e(t)$ which is a difference between reference signal $r(t)$ and array output $y(t)$:
  $$e(t) = r(t) - y(t) = r(t) - w^H x(t)$$

Cost function: $E[|e(t)|^2] = E[|r(t) - y(t)|^2]$

- $= E[|r(t) - w^H x(t)|^2]$ minimized
- $= E[|r(t)|^2] - w^T r_{xx}^* - w^H r_{xr} + w^H R_{xx} w$
- $r_{xr} = E[x(t)r^*(t)]$ (correlation vector)

MMSE adaptive array (Cont’d)

- Optimum weight vector
  $$w_{opt} = R_{xx}^{-1} r_{xr}$$

In the case of one desired signal incident

- $r_{xr} = \alpha v_0$
- ($\alpha$: constant)

This weight is equal to that of MSN adaptive array.

Maximizing output SINR
Example of 6-element adaptive array ($K = 6$)

- desired signal: $s(t)$, $P_s$, $\theta_s$
- interference: $u(t)$, $P_u$, $\theta_u$
- antenna elements: isotropic
- element spacing: $\lambda/2$
- MMSE: $r(t) = s(t)$
- MSN: $\theta_s$ is completely known

When $P_s = P_u = 1$, $P_t = 0.01$, $\theta_s = 30^\circ$, $\theta_u = -40^\circ$ (SNR=20dB, SIR=0dB)

Example of 6-element adaptive array ($K = 6$) (Cont’d)

**Adaptive** (MMSE & MSN)
- SINR = 27.66dB
- SINR$_0 = 0.9983$

**Uniform**
- SINR = 15.31dB
- SINR$_0 = 0.9714$
DOA estimation and array optimization

Beam-scan scheme

- **Beamformer**
  
  \[
  \max_w \left( \frac{1}{2} w^H a(\theta) a^H(\theta) w \right) \quad \text{subject to} \quad w^H w = 1
  \]

  \[
  P_{BF}(\theta) = w_{opt}^H R_{xx} w_{opt} = \frac{a^H(\theta) R_{xx} a(\theta)}{a^H(\theta) a(\theta)}
  \]

- **Capon Beamformer**
  
  \[
  \min_w \left( P_{out} = \frac{1}{2} w^H R_{xx} w \right) \quad \text{subject to} \quad w^H a(\theta) = 1
  \]

  \[
  P_{CP}(\theta) = w_{opt}^H R_{xx} w_{opt} = \frac{1}{a^H(\theta) R_{xx}^{-1} a(\theta)}
  \]
Beam-scan scheme
(Rayleigh quotient expression)

- Beamformer
  Mainbeam radiation efficiency
  \[
  \max_w \frac{w^H a(\theta) a^H(\theta) w}{w^H w} \quad \Rightarrow \quad w_{opt} = \alpha a(\theta)
  \]
  \(\alpha\): scalar

- Capon Beamformer
  SINR
  \[
  \max_w \frac{w^H a(\theta) a^H(\theta) w}{w^H R_{xx} w} \quad \Rightarrow \quad w_{opt} = \gamma R_{xx}^{-1} a(\theta)
  \]
  \(\gamma\): scalar

Problems of Beamformer

Wave 2 is received by sidelobe.
Closely spaced waves cannot be separated.
**Advantage of Capon Beamformer**

Examples of directional patterns by optimum weights

**How angular resolution is enhanced furthermore?**

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**Nulls** are utilized. Null-scan scheme

Nulls are adaptively steered in the respective directions of incident waves.

Angular spectrum of null-scan scheme:

$$P(\theta) = \frac{1}{|\mathbf{w}_{opt}^H \mathbf{a}(\theta)|^2}$$

Inverse of power pattern
Null-scan scheme (Constrained minimization)

- linearly constrained minimization (Linear prediction)

\[ \min_w \left( P_{out} - \frac{1}{2} w^H R_{xx} w \right) \text{ subject to } w^H u - 1 \]

\[ w_{opt} = R_{xx}^{-1} u \quad u = [1, 0, \cdots, 0]^T \]

- Quadratically constrained minimization

\[ \min_w \left( P_{out} = \frac{1}{2} w^H R_{xx} w \right) \text{ subject to } w^H w = 1 \]

\[ w_{opt} : \text{ eigenvectors corresponding to minimum eigenvalues of } R_{xx} \]

- Pisarenko’s method, Min-Norm method, MUSIC

Null-scan scheme (Rayleigh quotient expression)

- linearly constrained minimization (Linear prediction)

\[ \min_w \frac{w^H R_{xx} w}{w^H w} \quad \rightarrow \quad w_{opt} = R_{xx}^{-1} u \]

\[ u = [1, 0, \cdots, 0]^T \]

- Quadratically constrained minimization

\[ \min_w \frac{w^H R_{xx} w}{w^H w} \quad \rightarrow \quad w_{opt} = e_{min} \]

(Pisarenko’s method)
Eigenvectors of minimum eigenvalues of covariance matrix

eigenvalues when \( K \)-element array receives \( L(L < K) \) waves

\[
\lambda_1 \geq \cdots \geq \lambda_L > \lambda_{L+1} = \cdots = \lambda_K = \sigma^2
\]

Eigenvectors: \( e_1 \cdots e_L \)

\( e_{L+1} \cdots e_K \) Noise power

Example of 6-element ULA with element spacing of a half wavelength (\( L=3 \))

3 eigenvectors used individually

Some smoothing required

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Min-Norm method and MUSIC

**Min-Norm method**

\[
w = \alpha_{L+1} e_{L+1} + \cdots + \alpha_K e_K = E_N \alpha
\]

\[
\min_{\mathbf{w}} (\mathbf{w}^H \mathbf{w}) \text{ subject to } (E_S^H \mathbf{w} = 0 \text{ and } \mathbf{w}^H \mathbf{u} = 1)
\]

\( E_S = [e_1, \cdots, e_L], \ E_N = [e_{L+1}, \cdots, e_K], \ u = [1, 0, \cdots, 0]^T \)

Signal subspace Noise subspace \( \alpha = [\alpha_{L+1}, \cdots, \alpha_K]^T \)

\[
P_{MN}(\theta) = \frac{1}{|w_{opt}^H a(\theta)|^2} \times a^H(\theta) a(\theta)
\]

**MUSIC**

\[
P_{MUSIC}(\theta) = \frac{a^H(\theta) a(\theta)}{|e_{L+1}^H a(\theta)|^2 + \cdots + |e_K^H a(\theta)|^2}
\]

\[= \frac{a^H(\theta) a(\theta)}{a^H(\theta) E_N E_N^H a(\theta)} \]

Linear combination of eigenvectors

Combination of power patterns
Summary and conclusions

- Optimization of array antennas using eigenvalues and eigenvectors
  - Directivity
  - Adaptive arrays
  - DOA estimation
- Rayleigh quotient expression
  - Link to eigenvalue problem
  - Optimization of array antennas based on Rayleigh quotient expression

Unified array antenna theory including adaptive arrays and DOA estimation